Implementing CRA Interventions
Bradley Witzel

The Concrete to Representational to Abstract Sequence of Instruction

Bradley Witzel, Ph.D.
Winthrop University
witzelb@winthrop.edu
Twitter @BradWitzel

IES Practice Guide Recommendations

<table>
<thead>
<tr>
<th>#</th>
<th>IES - RtI Math Panel Recommendation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Screen ALL students to identify those who need interventions</td>
</tr>
<tr>
<td>2</td>
<td>Intervention instructional materials for students should focus on whole numbers (K-5) and rational numbers (4-8)</td>
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<tr>
<td>3</td>
<td>Intervention instruction should be explicit and systematic.</td>
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<td>4</td>
<td>Teach common underlying structures to word problems</td>
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<tr>
<td>5</td>
<td>Include visual representations of mathematical ideas</td>
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<tr>
<td>6</td>
<td>Devote at least 10 minutes to fluent fact retrieval</td>
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<tr>
<td>7</td>
<td>Monitor progress of those receiving intervention as well as those at-risk</td>
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<tr>
<td>8</td>
<td>Include motivational strategies in Tier 2 and Tier 3 interventions</td>
</tr>
</tbody>
</table>

USDOE on the Use of visual representations

- IES Practice Guide on RtI Math:
  - “Recommendation 5. Interventions should include opportunities for students to work with visual representations of mathematical ideas and interventions should be proficient in the use of visual representations of mathematical ideas” (2009).
- IES Practice Guide on Fractions:
  - “Recommendation 2. Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward” (2010)
- IES Practice Guide on Word Problems:
  - “Teach students how to use visual representations” (2012)
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CRA as effective instruction
(Gersten et al, 2009; NMP, 2008; Riccomini & Witzel, 2010; Witzel, 2005)

Concrete to Representational to Abstract Sequence of Instruction (CRA)
• Concrete (expeditious use of manipulatives)
• Representations (pictorial)
• Abstract procedures
Excellent for teaching accuracy and understanding
Example: http://engage.ucf.edu/v/p/2wKBsbB

CRA delivery
(From Witzel, Riccomini, & Schneider, 2008)
• Choose the math topic to be taught;
• Review procedures to solve the problem;
• Adjust the steps to eliminate notation or calculation tricks;
• Match the abstract steps with an appropriate concrete manipulative;
• Arrange concrete and representational lessons;
• Teach each concrete, representational, and abstract lesson to student mastery; and
• Help students generalize what they learn through word problems.

CRA
(Gersten et al, p. 32)

Solving the Equation with Concrete (Manipulatives) Stages and Tricks
• Stage 1: Spin around tricks
• Stage 2: Spin around tricks

Solving the Equation with Pictorial Stages

Solving the Equation with Abstract (Algebraic) Stages

Concrete Stages

A. One group of 5 equals 5; 4 more equals 9
B. Subtract 1 from each side of the equation
C. The equation now equals 1 group of 5 equals 8
D. One group of 5 equals 5; 1 more equals 6

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## Concrete Representational Abstract

<table>
<thead>
<tr>
<th>Concrete</th>
<th>Pictorial Representation</th>
<th>Abstract</th>
</tr>
</thead>
<tbody>
<tr>
<td>+ [image]</td>
<td>+</td>
<td>20 + 5</td>
</tr>
<tr>
<td>+ [image]</td>
<td>+</td>
<td>10 + 3</td>
</tr>
<tr>
<td>+ [image]</td>
<td>+</td>
<td>30 + 14</td>
</tr>
<tr>
<td>- [image]</td>
<td>-</td>
<td>44</td>
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</table>

(Witzel, et al., 2013)

### CCSS 1.NBT.4 Add within 100 using models and strategies based on place value

**26 + 18**

<table>
<thead>
<tr>
<th>Concrete</th>
<th>Pictorial Representation</th>
<th>Abstract</th>
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</thead>
<tbody>
<tr>
<td>+ [image]</td>
<td>+</td>
<td>26 + 18</td>
</tr>
</tbody>
</table>

(Witzel, et al., 2013)

### CCSS 2.NBT.7 Add and subtract within 100 using models and strategies based on place value

**33 - 18**

<table>
<thead>
<tr>
<th>Concrete</th>
<th>Pictorial Representation</th>
<th>Abstract</th>
</tr>
</thead>
<tbody>
<tr>
<td>+ [image]</td>
<td>+</td>
<td>33 - 18</td>
</tr>
</tbody>
</table>

(Witzel, et al., 2013)
Computation: Addition and Subtraction

While accuracy is important, it is the deliberate use of counting that should be assessed.

- Fifth grade “Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.”
- $7.6 \times 2.4 = ?$

<table>
<thead>
<tr>
<th>multiply</th>
<th>7</th>
<th>.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>14</td>
<td>1.2</td>
</tr>
<tr>
<td>.8</td>
<td>2.8</td>
<td>.24</td>
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</table>

$14 + 1.2 + 2.8 + 0.24 = 18.24$

Intervention Aspect of Place Value

- Language focused for conceptual building and to aid think alouds
- Physical models to show place value
  - http://ntnmath.com/video%20index/Index%20Videos/Polynomials/lesson%20108.htm
- Graphic Organizers for procedural memory and conceptual memory
- Possibility of a graduated sequence of instruction (CRA)
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(4.3)(2.4) using CRA

Ones times ones are ones. There are 8 ones.

Ones times tenths are tenths. There are 16 tenths.

Total = 8 ones; 22 tenths; 12 hundredths
8.0
2.2
0.12
10.32

Tenths times ones are tenths.
There are 6 tenths.

Tenths times tenths are hundredths.
There are 12 hundredths.

Total = 8 ones; 22 tenths; 12 hundredths
8.0
2.2
0.12
10.32

(4.3)(2.4) using CRA

multi | 4 | .3
---|---|---
2 | 8 | 0.6
.4 | 1.6 | 0.12

Total = 8 ones; 22 tenths; 12 hundredths
8.0
2.2
0.12
10.32
Physical Array Picture

Find new ways to reach the students

Fractions on a Number Line
(Witzel, REL-58, 2015)

“Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.”
(Siegler et al, 2010)

- Fraction Strips
- Fractional Clothesline
- Number Line computational practice
- Area to Number line connections
Standard Matches

4th
• MAFS.4.NF.1.1 Explain why a fraction a/b is equivalent to a fraction (n × a)/(n × b) by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.
• MAFS.4.NF.1.2 Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

5th
• MA.A.1.2.2: The student understands the relative size of whole numbers, commonly used fractions, decimals, and percents.
• MA.A.1.2.3: The student understands concrete and symbolic representations of whole numbers, fractions, decimals, and percents in real-world situations.

Fun with Fractions:
Making and Investigating Fraction Strips
(Witzel, REL-SE, 2015)

<table>
<thead>
<tr>
<th>1 whole</th>
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<tbody>
<tr>
<td>1/2</td>
</tr>
<tr>
<td>1/3</td>
</tr>
<tr>
<td>1/9</td>
</tr>
<tr>
<td>1/2</td>
</tr>
<tr>
<td>1/3</td>
</tr>
<tr>
<td>1/9</td>
</tr>
</tbody>
</table>

• Students cut, fold, and color strips of paper to create length-based models of fraction lines.
• Strips are stacked in order to make comparisons
• Ask questions such as, “Which strip is one-third of the whole?” and “Which strip is one-third of one-third?”

Fun with Fractions: Making and Investigating Fraction Strips (cont.)

• Use the term whole rather than one so that students understand the proportionality of fractions per a whole.
• In pairs, have students communicate relationships.
• List all relationships on chart paper and have students confirm or deny these relationships.

http://www.cpalms.org/Public/PreviewResource/Preview/30161
Fractional Clothesline

- Stretch a clothesline across the room.
- Pin cards to indicate location on a number line
- Vary cards between fractions, decimals, percents, and a combination
- Vary the objective from ordering to comparing
- Ask students to explain their reasoning

http://www.cpalms.org/Public/PreviewResourceUrl/Preview/5109

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Fractional Clothesline: Pinning Cards on the Line
(Witzel, REL-SE, 2015)

Variations include:
- Clothesline versus painter’s tape and sticky-notes
- Using key fraction benchmarks to assist students
- Graduating from whole class to small group or individual

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Fractional Clothesline: Extension
(Witzel, REL-SE, 2015)

Next Steps:
- a) fractions to decimals to percent
- b) combinations of cards
- c) change the representation of a whole

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Definition of Fraction

First define fractions with numerator 1 (unit fractions)

- \( \frac{1}{4} \) whole
- \( \frac{1}{4} \) to 1

Mixed to improper and back: concrete

- \( 8 \div 2 \)
- \( 8 \div 3 \)
- \( 13 \div 4 \)
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Intervention with Fractions procedures
[Witzel & Riccomini, 2009]

\[
\frac{2}{3} + \frac{1}{2} = \frac{2 + 1}{3 + 2} + \frac{7}{6}
\]

Aim interventions at procedural processes
[Witzel & Riccomini, 2009]

\[
\frac{1}{3} - \frac{2}{3} = \frac{1 + 1}{3 + 3} - \frac{2 + 2}{3 + 3}
\]

General Research Findings
(Gersten et al., 2009; Witzel, 2005; Witzel, in-press; Witzel, Mercer, & Miller, 2003)

- Higher effect sizes were for sequential use of representations (e.g., CRA) with clear and explicit stepwise consistency.
- When teachers used graphical representations only for demonstration, results were much less consistent. **Students must participate.**
- Visuals were not particularly useful unless students were provided opportunities to **practice** using them.
CRA Initiatives across the US

<table>
<thead>
<tr>
<th>Source</th>
<th>Representation</th>
<th>Value</th>
<th>Language</th>
</tr>
</thead>
</table>
| 243; 4 tens | 243; 4 tens | 243; 4 tens | Have any tens? What is the tens?
| 243; 3 ones | 243; 3 ones | 243; 3 ones | Have any ones? What is the ones?
| 200 + 60 + 3 (using place value mat) | 200 + 60 + 3 | 200 + 60 + 3 | What is the sum of the numbers in each column?

Example from PaTTAN, 2015

CRA is an approach to be used across grade or subject levels (Bruner, 1983)

Trigonometric ratios
Why would CRA be effective?
(Witzel, Riccomini, & Schneider, 2008)

• Multimodal forms of math acquisition to aid memory and retrieval
• Multiple learning styles are being met to aid relevance and motivation
• Meaningful manipulations of materials allow students to rationalize abstract mathematics
• Procedural accuracy; provides an alternative to algorithm memorization of math rules
• Transportable without concrete materials

Teach each CRA lesson to mastery

• Model and guide students in their use of manipulative objects and pictorial representations.
• Teach students step by step gradually introducing mathematical vocabulary. Allow students to name or invent their stepwise procedures within instruction.
• Move from concrete to representational to abstract learning levels only after students show accuracy without hesitations in manipulations or drawings.
• Assess each level of learning according to stepwise procedures. Take account of students who created different procedures.

Research Support

• From research studies
• To statewide initiatives
• To individual classrooms
Multisensory Algebra success

Research Support

- Statistic
  - Students with learning difficulties using this model outperformed peers on posttest and follow-up measures ($F=13.89, p<0.001$) [Witzel, Mercer, & Miller, 2003].
  - Students with a history of high math achievement scores also show benefit on the posttest ($F=10.37, p=0.01$) and the follow-up ($F=6.97, p=0.01$) despite pretest favoring of traditional ($F=12.18, p<0.001$) [Witzel, 2005].

- Testimonial
  - Teachers wanted to stop using their current instructional series and textbooks.
  - One teacher claimed that he would never teach algebra using any other method than through this model.

More studies are ongoing

Fractions Data (Witzel, in-review)

Tier 2 Fractions Research

6th Grade Students in 3 different states and 3 interventionists

CRA N=34; Abstract N=37

CRA outscored Abstract-only intervention students in the posttest and 6-week follow-up in adding and subtracting fractions with like and unlike denominators.
Choosing Manipulatives and Preparing for Instruction

- Review the abstract problem solving steps and processes before choosing manipulative objects
- Ask yourself:
  1. Are these manipulatives easy to use?
  2. How can these manipulatives be used for concept and process?
  3. Can I follow the same abstract steps using these manipulatives?
  4. How will the pictorial representation stage appear with these manipulatives?
  5. How many similar math skills can be taught using these manipulatives?

Next Steps with Visual Representations

- Working with manipulatives
  Manipulative objects do not teach children, teachers do. The manipulatives are mere tools to reach an outcome, usually an abstract one.
- Organize the sequence of the instructional steps (CRAMATH)
- Practice math dialogue to match instructional procedures.

Questions?

“Positive results can be achieved in a reasonable time at accessible cost, but a consistent, wise community-wide effort will be required.”

References